

The Modelling of Degenerate Neck Pinch Singularities in Ricci Flow by Bryant Solitons

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Abstract

In earlier work, carrying out numerical simulations of the Ricci flows of families of rotationally symmetric geometries on S^3 , we have found strong support for the contention that (at least in the rotationally symmetric case) the Ricci flow for a “critical” initial geometry—one which is at the transition point between initial geometries (on S^3) whose volume-normalized Ricci flows develop a singular neck pinch, and other initial geometries whose volume-normalized Ricci flows converge to a round sphere—evolves into a “degenerate neck pinch”. That is, we have seen in this earlier work that the Ricci flows for the critical geometries become locally cylindrical in a neighborhood of the initial pinching, and have the maximum amount of curvature at one or both of the poles. Here, we explore the behavior of these flows at the poles, and find strong support for the conjecture that the Bryant steady solitons accurately model this polar flow.

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1 Introduction

While a considerable amount has been learned during the past five years [1, 2, 3] [4, 5] about the Ricci flow of three dimensional Riemannian geometries, and while this knowledge has been used very successfully to study the relationship between topology and geometry [2, 6, 7, 8], there are still many unanswered questions regarding the details of three dimensional Ricci flow, especially concerning flows which develop singularities. One of these questions concerns the circumstances and the details of the formation of singularities of Type II. The Type II singularities are characterized by the condition that, if the singularity occurs at a finite time T , then the quantity $|Rm(t)|(T - t)|$ blows up as t approaches T ; this behavior contrasts with that of Type I singularities, for which this same quantity is uniformly bounded.

It is believed that the standard neck pinch singularities, which according to Hamilton's scenario [9] play a major role in three dimensional Ricci flow dynamics, are Type I. The work of Angenent and Knopf [10, 11] supports this contention, and describes some of the detailed asymptotic behavior of standard neck pinch singularities, at least in the rotationally symmetric case.

It has been conjectured [9], it has been demonstrated numerically [12], and it has now been proven [13], that Type II singularities occur during the course of a Ricci flow which uses a “critical geometry” for initial data. To obtain a critical geometry in the sense we mean here, one considers a one-parameter family of (initial data) Riemannian metrics on S^3 , with the volume-normalized Ricci flow developing standard neck pinch singularities for low values of the parameter, and with singularities avoided for those flows which start from geometries with high values of the parameter. The critical geometry then corresponds to the transitional (boundary) value of the parameter.

Our earlier work [12], in which we have numerically simulated the Ricci flow of critical geometries, uses families of geometries which are all rotationally symmetric and also reflection symmetric across the equator. The geometries are all “corsetted spheres”, with the parameter measuring the degree of corsetting relative to a round sphere. For tight corsetting (small parameter) the Ricci flow develops a standard neck pinch singularity, as described in [10]. For loose corsetting, the flow approaches the round sphere. In [12], we numerically simulate the Ricci flow of both tightly and loosely corsetted geometries, and we are then able to focus on the transitional critical geometry, and examine its flow.

The asymptotic behavior of the Ricci flow of the critical geometry seen in our simulations is quite different from that which we see starting at geometries with non critical values of the parameter. The curvature concentrates at the poles (recall our assumption of reflection symmetry), and overall the geometry approaches that of a “javelin”: it becomes increasingly cylindrical everywhere except at the poles, where the curvature blows up. In analogy with similar behavior seen for critical mean curvature flows, this has been labeled a “degenerate neck pinch”.

In our earlier work, we did not closely explore the details of the Ricci flow for our critical geometries at the poles. We noted the curvature blow up, but nothing further. Here, we focus on the flow at the poles, and we verify that, as has been conjectured, the flow in the neighborhood of the poles is very accurately modelled locally by the flow of the Bryant steady soliton [14]; see also chapter 1 of [5].

The Bryant steady soliton is the unique (up to scaling) rotationally symmetric Ricci gradient soliton on R^3 which neither shrinks nor expands. That is, the metric g on R^3 is rotationally symmetric about a fixed point on R^3 , taking the form

$$g = dr^2 + a(r)^2 \gamma_{\text{round}} \quad (1)$$

for the round metric γ_{round} on S^2 and for a positive function $a(r)$; and it satisfies the steady Ricci gradient soliton equation

$$R_{ab} + \nabla_a \nabla_b f = 0 \quad (2)$$

for a function f .

If one substitutes into equation (2) the metric g of the form (1) together with a rotationally symmetric function $f(r)$, one obtains a coupled system of ordinary differential equations, to be solved for $a(r)$ and $f(r)$. In [14], (See also chapter 1 of [5]) this ODE system is written out and analyzed, it is shown that there is a unique solution (up to homothety), and the profile of this solution (obtained by numerical integration) is exhibited. We use this profile in our model matching here. (See section 4.)

One of the key features of a steady Ricci soliton (g, f) is that the Ricci flow of the soliton metric g fixes g up to a time-dependent diffeomorphism (generated by the vector field $g^{-1}(df, \cdot)$). Hence the curvature is time independent. The sense in which such a flow can model the singularity developing at the pole of our degenerate neck pinch is in terms of the blow up (rescaling) of that singularity. Specifically, given a Ricci flow solution $g(t)$ which is going

singular at the point x^* at time T , let us define the “blow up” metrics $\tilde{g}(t)$ by setting

$$\tilde{g}(t) := \rho(t)g(t) \quad (3)$$

where the function $\rho(t)$ is chosen so that all the rescaled metrics $\tilde{g}(t)$ have the same value of $|Rm|$ at the point x^* . The singularity is then modeled by the soliton if, for spatial points near x^* and for t approaching T , the blow up metrics \tilde{g} approach closer and closer to the soliton metric. We provide strong numerical evidence here that for the degenerate neck pinch solutions which we study here, the geometry at the poles is indeed modelled by Bryant steady solitons in this sense.

Since, in any family of initial geometries, it is difficult to precisely find the critical initial geometry which flows (via Ricci flow) to a degenerate neck pinch, for our numerical studies we have chosen to carry out a slightly different comparison as well: we consider a sequence of initial geometries g_α which are all sub critical (loose corsetting) and which approach the critical geometry. Since these are sub critical, each of their flows evolve toward a time t_α (different for each one) at which the curvature at the poles reaches a maximum, after which their flows dissipate the curvature. So one alternative way to test the Bryant steady soliton modelling conjecture is to evolve each of the g_α geometries up to its time of maximum curvature t_α and then compare the geometry near the pole of $g_\alpha(t_\alpha)$ with the Bryant steady soliton, scaled to have the same maximum geometry as $g_\alpha(t_\alpha)$. Our numerical work indicates that this comparison too matches very well.

When our initial work [12] on the numerical simulation of the Ricci flow of degenerate neck pinches was carried out, it had not been shown mathematically that Type II singularities do develop during Ricci flow. This has now been shown by Gu and Zhu [13]. Their work does not, however, tell us any of the details of such Ricci flows. Our work here indicates what those details should be. One hopes that a proof of this behavior will be forthcoming.

2 Ricci flow

The types of metrics considered are the same as in [12]. We have spherically symmetric metrics on S^3 which take the form

$$ds^2 = e^{2X} \left(e^{-2W} d\psi^2 + e^{2W} \sin^2 \psi [d\theta^2 + \sin^2 \theta d\phi^2] \right). \quad (4)$$

The corseted sphere geometries that we use for initial data have $W = X$ with X determined intrinsically by

$$\begin{aligned} 4e^{4X}\sin^2\psi &= \sin^2 2\psi \quad \text{for } \cos^2\psi \geq \frac{1}{2} \\ 4e^{4X}\sin^2\psi &= \sin^2 2\psi + 4\lambda \cos^2 2\psi \quad \text{for } \cos^2\psi \leq \frac{1}{2}, \end{aligned} \quad (5)$$

where λ is a constant that determines the amount of corsetting. In particular $\lambda = 0$ corresponds to two round three spheres joined at the poles, while for positive λ this cusp is smoothed out.

To ensure that the evolution is well behaved numerically, we evolve using the volume-normalized DeTurck flow [15]

$$\partial_t g_{ab} = -2R_{ab} + 2D_{(a}V_{b)} + \frac{2}{3}\hat{r}g_{ab} \quad (6)$$

Here the spatial constant \hat{r} is the volume average of the scalar curvature, and the vector field V^a is given by

$$V^a = g^{bc}(\Gamma_{bc}^a - \Delta_{bc}^a), \quad (7)$$

where Γ_{bc}^a is the connection of the metric g_{ab} while Δ_{bc}^a is the connection of a round three sphere. This evolution equation for the metric g , together with the form of g in (4), yields partial differential equations for the metric quantities X and W . However, we find that it is more convenient to use X and the quantity $S \equiv W/\sin^2\psi$. The reason for this is that smoothness of the metric requires that W vanish at the poles of the three sphere at the rate of $\sin^2\psi$ and this condition is automatically enforced by smoothness of S . The evolution equations for X and S that follow from equation (6) are

$$\begin{aligned} \partial_t X &= e^{2(W-X)} \left[X'' + 2\cot\psi X' - 2 + \frac{1}{2}([X']^2 + [W']^2) + 3X'W' \right. \\ &\quad \left. + (1 - e^{-4W}) \left(\frac{1}{2\sin^2\psi} + 1 + 2\cot\psi W' \right) \right] + \frac{\hat{r}}{3}. \end{aligned} \quad (8)$$

$$\begin{aligned} \partial_t S &= e^{2(W-X)} \left[S'' + 6\cot\psi S' - 8S - \frac{3}{2\sin^4\psi} (1 - 4W - e^{-4W}) \right. \\ &\quad \left. + \frac{1 - e^{-4W}}{\sin^2\psi} (1 - 2[\cot\psi X' + 2\sin\psi \cos\psi S' + 4\cos^2\psi S]) \right. \\ &\quad \left. - \frac{1}{2}([X'/\sin\psi]^2 + [\sin\psi S' + 2\cos\psi S]^2 \right. \\ &\quad \left. + 6[X'/\sin\psi][\sin\psi S' + 2\cos\psi S]) \right]. \end{aligned} \quad (9)$$

Here a prime denotes differentiation with respect to ψ and the quantities W and W' should be thought of as derived from S through $W = S\sin^2\psi$ and $W' = S'\sin^2\psi + 2S\sin\psi\cos\psi$. The quantity \hat{r} is calculated by

$$\hat{r} = \frac{2}{N} \int_0^\pi d\psi e^{X+3W} \left(e^{-4W} - 1 - 4\sin\psi\cos\psi W' + \sin^2\psi[3 + (X' + W')^2] \right), \quad (10)$$

where the normalization constant N is given by

$$N \equiv \int_0^\pi d\psi e^{3X+W} \sin^2\psi. \quad (11)$$

3 Numerical methods

To numerically integrate the PDEs for X and S , we approximate these functions by their values on a grid and approximate the PDEs by finite difference equations. Let F stand for the pair (X, S) and define the numbers F_i^n by

$$F_i^n = F((i - \frac{3}{2})\Delta\psi, n\Delta t) \quad (12)$$

That is, the F_i^n are the values of F on a grid with spatial step $\Delta\psi$ and time step Δt . For a PDE of the form $\partial_t F = \mathcal{O}F$ for some operator \mathcal{O} we approximate \mathcal{O} by the finite difference operator $\hat{\mathcal{O}}$ given by replacing all spatial derivatives in \mathcal{O} with centered finite differences. We then approximate $\partial_t F$ by $(F_i^{n+1} - F_i^n)/\Delta t$. Since at any given time step we know F_i^n and want to solve for F_i^{n+1} the simplest thing to do is to apply $\hat{\mathcal{O}}$ at time step n , which yields the finite difference equation

$$\frac{F_i^{n+1} - F_i^n}{\Delta t} = \hat{\mathcal{O}}F_i^n, \quad (13)$$

which has the solution

$$F_i^{n+1} = F_i^n + \Delta t \hat{\mathcal{O}}F_i^n. \quad (14)$$

This is the method used in the work described in [12]. Unfortunately, this method is quite slow for the following reason: as is typical for parabolic equations, stability of the numerical method requires a time step Δt that is of order $(\Delta\psi)^2$. This means that it becomes extremely slow to run simulations with high resolution. For this reason—limitations in resolution—we were able

in [12] to show the existence of a critical solution, but we were not able to examine that critical solution accurately enough to characterize it. To obtain the needed level of accuracy without sacrificing efficiency, we need a faster numerical method—one that does not have such a limited time step. One way to overcome the limitation on the time step is to apply the operator $\hat{\mathcal{O}}$ at time step $n + 1$ rather than time step n . This yields

$$F_i^{n+1} = F_i^n + \Delta t \hat{\mathcal{O}} F_i^{n+1}. \quad (15)$$

At first this equation does not seem helpful, since we are given F_i^n and we want to find F_i^{n+1} . However, this equation has the solution

$$F_i^{n+1} = [I - \Delta t \hat{\mathcal{O}}]^{-1} (F_i^n), \quad (16)$$

where I is the identity operator. This equation has the advantage that stability places no restriction on the size of the time step. It also has the disadvantage that it requires the inversion of the nonlinear operator $I - \Delta t \hat{\mathcal{O}}$ a process that is both difficult and slow. However, it is only the principal part of the operator $\hat{\mathcal{O}}$ that leads to the restriction to small time step. The solution to this dilemma is then to split $\hat{\mathcal{O}}$ into two parts, one of which contains the principal part but is also simple enough to be inverted quickly. We then apply that part at time step $n + 1$ and the rest at time step n . Specifically, we write the equations of motion as

$$\hat{\mathcal{O}}_1(X_i^{n+1}) = \hat{\mathcal{O}}_3(X_i^n, S_i^n) \quad (17)$$

$$\hat{\mathcal{O}}_2(S_i^{n+1}) = \hat{\mathcal{O}}_4(X_i^n, S_i^n), \quad (18)$$

where the operators above are the finite difference version of

$$\mathcal{O}_1(X) = \frac{e^{2(X-W)}}{\Delta t} X - (X'' + 2 \cot \psi X') \quad (19)$$

$$\mathcal{O}_2(S) = \frac{e^{2(X-W)}}{\Delta t} S - (S'' + 6 \cot \psi S') \quad (20)$$

$$\begin{aligned} \mathcal{O}_3(X, S) = & \frac{e^{2(X-W)}}{\Delta t} X - 2 + \frac{1}{2} ([X']^2 + [W']^2) + 3X'W' \\ & + (1 - e^{-4W}) \left(\frac{1}{2\sin^2 \psi} + 1 + 2 \cot \psi W' \right) + \frac{\hat{r}}{3} e^{2(X-W)} \end{aligned} \quad (21)$$

$$\mathcal{O}_4(X, S) = \frac{e^{2(X-W)}}{\Delta t} S - 8S - \frac{3}{2\sin^4 \psi} (1 - 4W - e^{-4W})$$

$$\begin{aligned}
& + \frac{1 - e^{-4W}}{\sin^2 \psi} \left(1 - 2[\cot \psi X' + 2 \sin \psi \cos \psi S' + 4 \cos^2 \psi S] \right) \\
& - \frac{1}{2} ([X'/\sin \psi]^2 + [\sin \psi S' + 2 \cos \psi S]^2 \\
& + 6[X'/\sin \psi][\sin \psi S' + 2 \cos \psi S]). \tag{22}
\end{aligned}$$

In the expressions for \mathcal{O}_1 and \mathcal{O}_2 the quantity $e^{2(X-W)}$ is evaluated at time step n even though the operator is applied to an argument at time step $n+1$. Thus given the metric functions (X, S) at time step n we produce their values at time step $n+1$ through

$$X_i^{n+1} = \hat{\mathcal{O}}_1^{-1}(\hat{\mathcal{O}}_3(X_i^n, S_i^n)) \tag{23}$$

$$S_i^{n+1} = \hat{\mathcal{O}}_2^{-1}(\hat{\mathcal{O}}_4(X_i^n, S_i^n)). \tag{24}$$

The operators $\hat{\mathcal{O}}_1$ and $\hat{\mathcal{O}}_2$ are linear operators that are easily and rapidly inverted using the cyclic tridiagonal method as given in [16]

4 Results and comparison with the Bryant steady soliton

All runs of the computer code have been carried out in double precision with 10,000 spatial grid points and with $dt = d\psi$. Through a binary search, we have determined the critical value of λ and have then examined the behavior of the curvature for several slightly subcritical solutions. In all cases, during the course of Ricci flow, the curvature becomes large and then diminishes, with the maximum of the curvature occurring at the poles. Figure (1) shows the scalar curvature at the pole as a function of time for three different subcritical solutions.

It is clear from this figure that as the initial data gets closer to the critical data, the maximum curvature occurring during the corresponding flow gets larger. This suggests that to get the best view (via numerical simulation) of the critical solution, one should examine the evolving metric at the time for which the curvature of a slightly subcritical solution is at its maximum.

As noted in the introduction, we follow two approaches here to see if the behavior of the critical solution at the poles is modeled accurately by the Bryant steady soliton. Before carrying either of them through, we need a numerical simulation of the metric for this soliton. To obtain this, we first

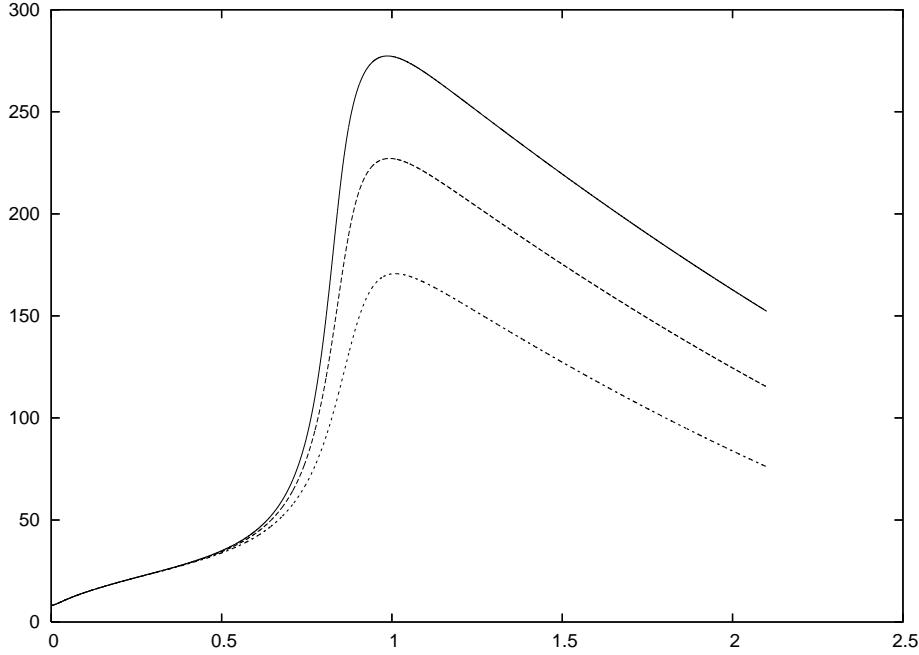


Figure 1: Plot of scalar curvature at the pole *vs* time for three different values of λ

substitute a metric of the form (1) into the Ricci soliton equation (2), and we are led to the following ODE system

$$f'' = \frac{2a''}{a} \quad (25)$$

$$a'' = f'a' + \frac{1 - (a')^2}{a}, \quad (26)$$

where a prime denotes differentiation with respect to r . Since this is a system of second order ODEs for two functions, one might expect a four parameter family of solutions. However, equations (25)-(26) are invariant under the addition of a constant to f and smoothness of the solution at the origin imposes additional constraints. In the end, the general smooth solution is determined by a single parameter α which appears in the series expansions for a and f' around the origin as follows

$$a = r + \alpha r^3 + \dots \quad (27)$$

$$f' = 12\alpha r + \dots \quad (28)$$

where the dots \dots stand for higher order terms. However, even this degree of freedom is to a certain extent misleading. It is only for negative α that nonsingular solutions exist for $0 \leq r < \infty$ and solutions with different negative values of α differ only in an overall constant scale factor in the metric. Thus up to scale, there is really only one Bryant steady soliton. The relation between the overall scale and the parameter α is reflected in the expression $R = -36\alpha$ for the scalar curvature at the origin.

To develop the Bryant steady soliton using (25) and (26), we first choose α (the choice depending on the desired scale; see below) and use it to express the expansion forms for a and f' in a neighborhood of the origin, following (27) and (28). We then numerically integrate the rest of the way using the fourth order Runge-Kutta method.[16]

For a spherically symmetric metric, the Ricci tensor has two independent eigenvalues: R_{S^2} the eigenvalue in the directions tangent to the symmetry S^2 and R_\perp the eigenvalue in the direction perpendicular to the symmetry S^2 . For both our Ricci flow simulations and for our numerical integration of the Bryant soliton ODEs, we calculate these Ricci eigenvalues and compare them.

For one of our comparison studies, we compare the geometry of appropriately scaled Bryant steady soliton solutions with the geometries near the poles at the time of maximum curvature for a sequence of subcritical initial geometries approaching the critical solution. Here, we choose the parameter α so that the curvature at the tip of the soliton matches that at the poles of the flows for the sub critical geometries (at maximum curvature). Figure (2) gives the R_\perp eigenvalue for the Ricci flow simulation at the time of maximum curvature (solid line) and for the Bryant soliton (dashed line). These quantities are plotted as functions of radial distance. Figure (3) makes the same comparison for the R_{S^2} eigenvalue. Note that in both cases there is an excellent match between the Ricci flow simulation and the Bryant soliton.

For our other study, we choose a subcritical initial metric which is very close to the critical geometry, and consider a sequence of times approaching the time of maximum curvature at the pole. At each time, we calculate the blow up geometry as specified in equation 3. We scale the blowups, and scale the soliton, so that they all have identical curvature at the poles. The results are graphed in Figure (4) for R_\perp and in Figure (5) for R_{S^2} . In these simulations, the maximum curvature occurs at a time of approximately 1.0.

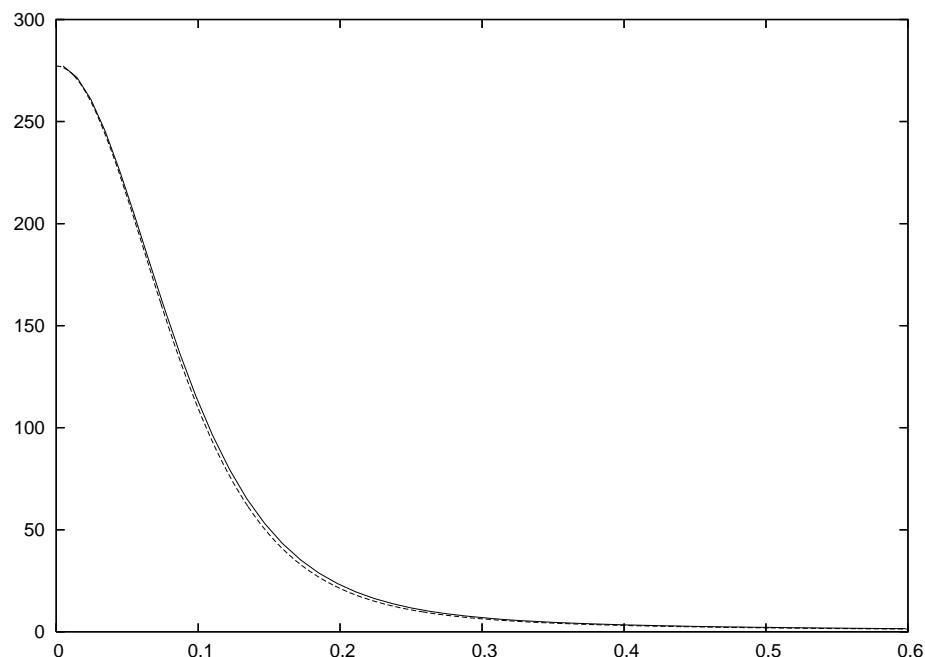


Figure 2: R_{\perp} as a function of radial length for a near critical solution at the time of maximum curvature (solid line) and for the Bryant soliton (dashed line)

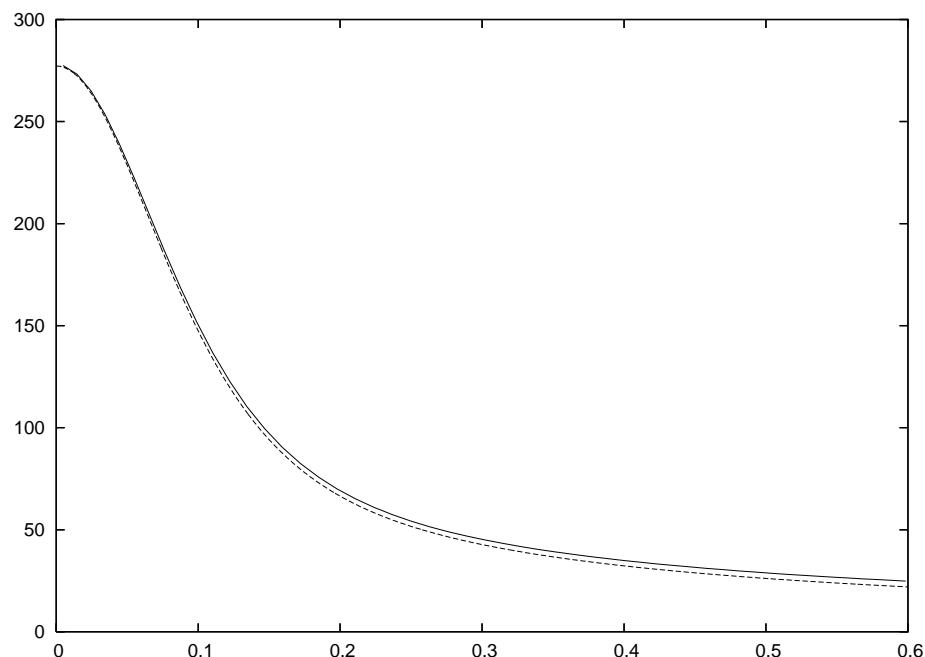


Figure 3: R_{S^2} as a function of radial length for a near critical solution at the time of maximum curvature (solid line) and for the Bryant soliton (dashed line)

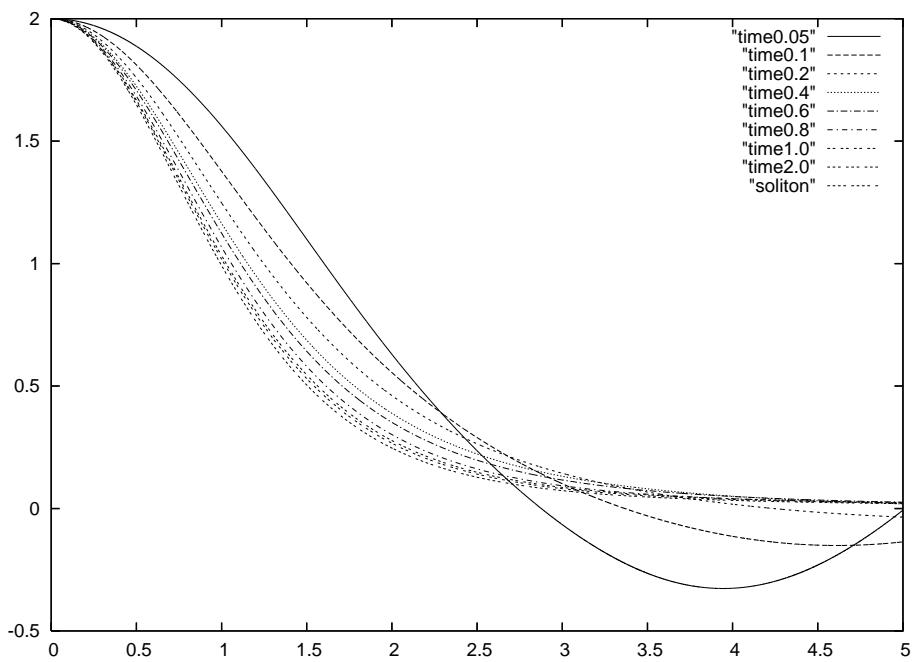


Figure 4: Plot of R_{perp} as a function of radial length both for the rescaled metrics of several different times and for the Bryant steady soliton

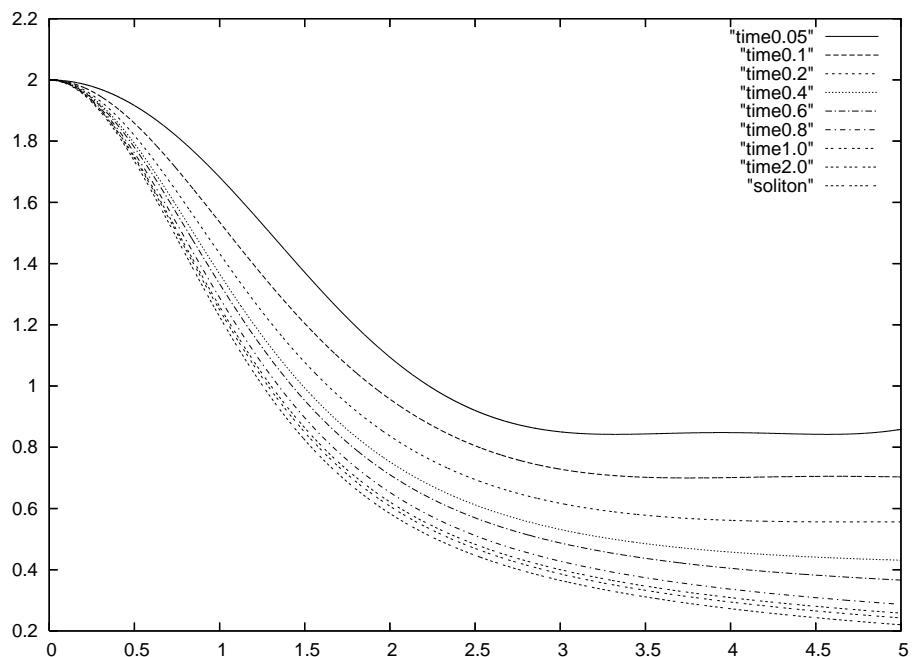


Figure 5: Plot of R_{S^2} as a function of radial length both for the rescaled metrics of several different times and for the Bryant steady soliton

We see that the blowup geometries approach agreement with the Bryant steady soliton at somewhat early times and retain this agreement for times beyond the time of maximum curvature. This agreement (in a neighborhood of the pole) becomes remarkably close. Thus both of our tests strongly support the conjecture that the Bryant steady soliton accurately models the pole behavior of the Ricci flow of critical geometries which develop degenerate singularities.

5 Conclusions

Since for each one parameter family of initial geometries one expects degenerate pinches to occur for just a single value of the parameter, direct numerical testing of the behavior of degenerate neck pinches is essentially impossible. Our numerical studies here do, however, strongly support the contention that degenerate neck pinches are modeled very accurately by the Bryant steady soliton, at least in the case of rotationally symmetric geometries.

The natural next step for these studies is to consider one parameter families of geometries which are *not* rotationally symmetric. Numerical simulation of the Ricci flow for such metrics is expected to be considerably more challenging, and likely will require working on multiple overlapping patches on S^3 . Carrying out these simulations, however, should allow us to explore whether the Ricci flows of non rotationally symmetric geometries tend to evolve toward rotationally symmetric geometries, both in the case of neck pinch singularity formation, and degenerate neck pinch singularity formation.

Once numerical evidence for a particular behavior in solutions of a PDE system has been obtained, there is strong motivation to mathematically prove that the behavior is present. As noted above, the existence of Type II singularities in the Ricci flows of critical type geometries has now been proven [13]. However, the features of these singularities, including the formation of javelin geometries and the modeling by Bryant steady solitons, remains mathematically unverified. This should be a promising direction for future research.

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